Equilibrium Enthalpy Profiles for the Incompressible Turbulent Boundary Layer with Heat Transfer

IRWIN E. ALBER*
Northrop Corporate Laboratories, Hawthorne, Calif.

AND

DOUGLAS E. COATS†
Dynamic Science, Irvine, Calif.

Incompressible enthalpy profiles are calculated from the thermal energy equation for a family of equilibrium (or similar) turbulent boundary layers subject to a prescribed pressure gradient parameter β_T or wall temperature gradient parameter α_T. The wall and wake eddy conductivity constants are determined by a fit of calculated and measured constant pressure, constant wall temperature enthalpy profiles. It is shown that the Reynolds analogy parameter for heat transfer and the turbulent boundary layer enthalpy profiles are adequately described by the analysis when wall pressure and temperature gradients are nonzero.

Nomenclature

B_h, B = empirical constants in the enthalpy and velocity law of the wall, see Eqs. (11) and (13)
C_f = wall skin friction coefficient
C_H = $q_w/\rho u_e(h_w - h_e)$, Stanton number
γ = $(C_f/2)^{1/2}$
G = $\int_0^\infty [(u_e - u)/u_\tau]^2 dy/\Delta$, Clauser shape parameter
h = static enthalpy
h_τ = $-q_w/\rho u_\tau$, heat transfer enthalpy
H = total enthalpy
K_h = outer layer constant for ϵ
q = heat flux
R_{δ^*} = Reynolds number based on δ^*
$\mathcal{R}$ = $2C_H/C_f$, Reynolds analogy factor
S = $(h_e - h)/h_\tau$ enthalpy defect variable
u,v = components of fluid velocity in direction of increasing x,y, respectively
u_τ = $(\tau_w/\rho)^{1/2}$, friction velocity
x,y = coordinates parallel and normal to flow direction
α_T = $\rho u_e \delta^*(dh_w/dx)/q_w$, turbulent equilibrium wall enthalpy parameter
β_T = $-\rho u_e \delta^*(du_e/dx)/\tau_w$, turbulent equilibrium pressure gradient parameter
δ^* = displacement thickness
ϵ_h = eddy conductivity
Δ = $\int_0^\infty [(u_e - u)/u_\tau] dy$
$\varkappa, \varkappa_h$ = velocity and enthalpy Karman constant, see Eqs. (11) and (13)
η = similarity variable y/Δ
ν = kinematic viscosity
ρ = fluid density
τ = shear stress

Subscripts

$(\ )_e$ = boundary layer edge
$(\ )_w$ = wall

Superscripts

$(\ )'$ = differentiation with respect to η
$\langle\ \rangle$ = averaged fluctuation quantity

Presented as Paper 69-689 at the AIAA Fluid and Plasma Dynamics Conference, San Francisco, Calif., June 16–18, 1969; submitted December 1, 1969; revision received September 21, 1970. This work was sponsored by Project SQUID which is supported by the Office of Naval Research, Department of the Navy, under Contract N00014-67-A-0026-0005, NR-098-038.

* Senior Staff Scientist, Continuum Mechanics Laboratory. Member AIAA.

† Member of the Technical Staff, Computer Sciences Department.

Introduction

SEVERAL experimentalists[1,2] have recently presented a considerable amount of new data for heat transfer rates and temperature distributions in supersonic and hypersonic turbulent boundary layers. They conclude that the Crocco relation, usually employed in many compressible methods, provides a good approximation to actual temperature distributions only for the case of flat plate flows ($dP/dx \approx 0$). Measured temperature distributions for nozzle wall boundary layers with heat transfer were found to display large deviations from the Crocco solution. It was noted that the temperature distributions of boundary layers on nozzle walls not only differ greatly from the Crocco solution but are not representative of those measured for flat plates. It is therefore concluded that Crocco's relation cannot be applied to boundary layers with pressure gradients and heat transfer.

To gain some idea as to the manner in which high-speed enthalpy profiles and heat transfer coefficients are affected by pressure gradients, and wall temperature gradients, an analysis using an appropriate eddy conductivity model is presented below to determine the family of equilibrium (or similar) enthalpy profiles for the more readily solvable case of turbulent incompressible flow. We employ the approach that Mellor and Gibson[3] adopted in deducing the general family of incompressible equilibrium velocity profiles. In Ref. 3, equilibrium velocity profile solutions were obtained in the form

$$(u_e - u)/u_\tau = f'(\eta), \quad \eta = y/\Delta \tag{1}$$

for flows where the pressure gradient parameter β_T = const.

By considering equilibrium flows, a simple closed form expression for the Reynolds analogy factor $\mathcal{R} = 2C_H/C_f$ [see Eq. (16)] is derived herein by matching the enthalpy law of the wall to the enthalpy defect solution. The effects of pressure gradient and wall temperature gradient are easily determined by use of the tabulated constants of integration obtained from the numerical solution of the energy equation. In the present work it is not necessary to employ the approximations made in the previous analyses of Hudumoto[4] or

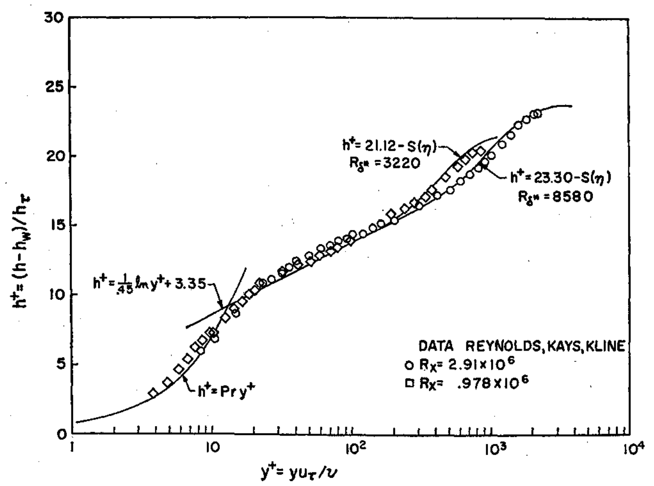

Fig. 1 Incompressible turbulent flat plate enthalpy profile in law of the wall coordinates. Comparison of similarity solutions (solid line) with data of Ref. 8.

Tetervin,[5] such as polynomial representations for τ and q, and the assumption of equal thermal and energy thicknesses. In addition to the approximate analyses of Refs. 4 and 5 there exist several finite difference computer codes in current use which make it possible to calculate turbulent enthalpy profiles and heat transfer coefficients (given an expression for ϵ and ϵ_h) for arbitary wall pressure and temperature distributions (e.g., Cebeci et al.[6] and Herring and Mellor[7]). However, the results from such calculations do not as readily lend themselves to the broad interpretation that is provided by a general family of equilibrium profiles. It is in this spirit of providing more insight into the physical processes of turbulent flows with heat transfer that the following analysis is presented.

Boundary-Layer Equations

The boundary layer form of the energy equation may be written as

$$u\partial h/\partial x + v\partial h/\partial y = -(1/\rho)\partial q/\partial y \tag{2}$$

with the turbulent heat flux, q, represented by an eddy conductivity formulation,

$$q = \rho\langle v'h'\rangle = -\rho\epsilon_h(\partial h/\partial y) \tag{3}$$

The boundary conditions for Eq. (1) are taken to be the following:

$$\lim_{y\to\infty} (h_e - h) = 0 \tag{4}$$

$$\lim_{y\to 0} \rho\epsilon_h(\partial h/\partial y) = -q_w \tag{5}$$

The dissipation and pressure work terms are omitted from the energy equation as it is assumed for low-speed flow that $(\gamma - 1)M_e^2 \ll (h_w - h_e)/h_e$.

Following the formulation of Mellor and Gibson for the turbulent eddy viscosity, the eddy conductivity distribution is broken up into an outer layer diffusivity of the form, $\epsilon_h = K_h u_e \delta^*$ (outer layer), and an overlap or wall layer diffusivity based on Prandtl's mixing length theory, $\epsilon_h = \varkappa\varkappa_h y^2|\partial u/\partial y|$ (overlap layer). The dividing point between the inner and outer thermal layer is defined as the value of y where $\varkappa\varkappa_h y^2|\partial u/\partial y| = K_h u_e \delta^*$. The constants for ϵ_h are to be based on a best fit of calculated and measured flat plate enthalpy profiles.

Enthalpy Defect Solutions

Defect solutions of the energy equation, Eq. (2), are sought of the form

$$(h_e - h)/h_\tau = S(\eta), \quad h_\tau \equiv -q_w/(\tau_w\rho)^{1/2} \tag{6}$$

The u and v components of velocity are determined respectively from Eq. (1) and the continuity equation, i.e.,

$$u = u_e(1 - f'\gamma), \quad \gamma \equiv u_\tau/u_e \tag{7}$$

$$v = -(\eta - f\gamma)\Delta du_e/dx + u_e f\Delta d\gamma/dx - u_e\gamma(d\Delta/dx)(\eta f' - f) \tag{8}$$

Substituting Eqs. (6–8) into Eq. (2) and defining a nondimensional thermal diffusivity, $\Phi_h = \epsilon_h/u_e\delta^*$, we obtain after algebraic manipulation, the governing ordinary differential equation for the equilibrium enthalpy profiles:

$$[fa + (f - \eta/\gamma)b]S' + (1/\gamma - f')cS = (\Phi_h S')' \tag{9}$$

subject to the boundary conditions

$$\lim_{\eta\to\infty} S = 0, \quad \lim_{\eta\to\infty} \Phi_h S' = -1 \tag{10}$$

The coefficients a, b, and c, are defined as follows:

$$a = (\Delta/\gamma)d\gamma/dx, \quad b = (1/u_e)(d/dx)(u_e\Delta)$$

$$c = (\Delta/\gamma_h)d\gamma_h/dx, \quad \gamma_h = h_\tau/h_e$$

Mellor and Gibson have derived expressions for a and b in terms of γ and β_T by making use of the integral momentum equation and the following local friction law:

$$1/\gamma = (1/\varkappa)\ln(\Delta u_\tau/\nu) + A(\beta) + B \tag{11}$$

From those equations it is found that

$$a = (\gamma/\kappa)(\gamma\beta_T)[1 + 1/m]/(1 + \gamma/\kappa)$$
$$b = -\gamma\beta_T[1 + 1/m] \tag{12}$$

Table 1 Tabulated equilibrium solutions of the incompressible turbulent energy equation

			$[\alpha_T = 0]$				
β_T	−0.4	0.0	0.5	1.0	2.0	4.0	10.0
A_h	8.59	−0.17	−3.08	−4.59	−6.34	−8.21	−10.76
I_1,[a]	13.25	6.35	4.56	3.78	3.00	2.33	1.64
I_2,[b]	3.51	0.99	0.55	0.38	0.24	0.15	0.07
η	S	S	S	S	S	S	S
0.0001	29.06	20.29	17.39	15.88	14.12	12.25	9.71
0.0002	27.52	18.75	15.85	14.34	12.59	10.73	8.22
0.0005	25.48	16.72	13.81	12.31	10.58	8.74	6.33
0.0010	23.93	15.18	12.28	10.79	9.07	7.28	5.00
0.0020	22.38	13.64	10.76	9.28	7.61	5.89	3.82
0.0050	20.32	11.60	8.77	7.34	5.76	4.22	2.55
0.0100	18.73	10.06	7.30	5.94	4.48	3.15	1.82
0.0200	17.11	8.54	5.89	4.64	3.36	2.27	1.25
0.0300	16.13	7.65	5.10	3.94	2.79	1.80	0.83
0.0400	15.41	7.01	4.56	3.44	2.32	1.37	0.50
0.0500	14.84	6.50	4.07	2.97	1.89	1.00	0.27
0.0600	14.30	5.98	3.59	2.53	1.50	0.70	0.13
0.0700	13.75	5.49	3.14	2.12	1.17	0.47	0.05
0.0800	13.22	5.01	2.72	1.75	0.88	0.30	0.02
0.0900	12.68	4.55	2.34	1.43	0.65	0.18	0.01
0.1000	12.16	4.11	1.98	1.14	0.46	0.10	0.00
0.1200	11.13	3.30	1.38	0.70	0.22	0.03	
0.1400	10.13	2.59	0.92	0.40	0.09	0.01	
0.1600	9.18	1.99	0.59	0.21	0.03	0.00	
0.1800	8.26	1.49	0.36	0.10	0.01		
0.2000	7.41	1.09	0.21	0.05	0.00		
0.2500	5.48	0.45	0.04	0.00			
0.3000	3.91	0.16	0.01				
0.3500	2.68	0.05	0.00				
0.4000	1.77	0.01					
0.4500	1.12	0.00					
0.5000	0.68						
0.6000	0.22						
0.7000	0.06						
0.8000	0.01						
0.9000	0.00						

[a] $I_1 = \int_0^\infty Sf'd\eta$.
[b] $I_2 = \int_0^\infty Sd\eta$.

where

$$\beta_T[1 + 1/m] = -(1 + \gamma/\varkappa)[1 + 2\beta_T - \gamma G\beta_T]/[1 - \gamma G + (\gamma^2 G/\varkappa)]$$

In order to determine an expression for the parameter c (the rate of change of the heat flux coefficient, γ_h) it is necessary to determine an expression for a local heat flux law similar to Eq. (11). The enthalpy distribution in the turbulent wall region can be obtained by integrating Eq. (3) from the edge of the laminar sublayer assuming that $q = q_w =$ constant, $\epsilon_h = \varkappa\varkappa_h y^2|\partial u/\partial y|$ and $\partial u/\partial y = u_\tau/\varkappa y$; thus yielding

$$(h - h_w)/h_\tau = (1/\varkappa_h)\ln(yu_\tau/\nu) + B_h \quad (13)$$

By matching Eq. (13) with the constant pressure, constant wall temperature data of Reynolds, Kays, and Kline[4] in the turbulent wall region (see Fig. 1), the unknown constants appearing therein are found to be $\varkappa_h = 0.45$, $B_h = 3.35$. The deviation in the wake region is contained in the defect solution of Eq. (9).

For small η, the defect solution has the form

$$S = -(1/\varkappa_h)\ln\eta + A_h \quad (14)$$

If we now assume the existence of an overlap region where h is described equally well by Eqs. (13) and (14), these equations may be added to give the local heat transfer law

$$(h_e - h_w)/h_\tau = (1/\varkappa_h)\ln(\Delta u_\tau/\nu) + A_h + B_h \quad (15)$$

Comparing Eqs. (15) and (11) we find the following relation between the Stanton number, $C_H = q_w/\rho u_e(h_w - h_e)$, and the skin friction coefficient, C_f, which may be thought of as a modified form of Reynolds analogy

$$1/\mathcal{R} = C_f/2C_H = \varkappa/\varkappa_h + (C_f/2)^{1/2}[(A_h - \varkappa A/\varkappa_h) + (B_h - \varkappa B/\varkappa_h)] \quad (16)$$

Table 2 Tabulated equilibrium solutions of the incompressible turbulent energy equation

			$[\beta_T = 0]$			
α_T	−3.0	−2.0	−1.0	0.5	1.5	2.5
A_h	−6.52	−5.54	−3.90	3.93	20.03	41.82
I_1	2.05	2.60	3.66	9.49	22.22	39.61
I_2	0.24	0.33	0.51	1.59	4.08	7.50
η	S	S	S	S	S	S
0.0001	13.95	14.92	16.57	24.40	40.49	62.29
0.0002	12.41	13.39	15.03	22.85	38.95	60.74
0.0005	10.39	11.36	13.00	20.82	36.91	58.69
0.0010	8.87	9.83	11.47	19.27	35.35	57.11
0.0020	7.38	8.33	9.94	17.72	33.77	55.51
0.0050	5.46	6.38	7.96	15.66	31.62	53.25
0.0100	4.11	4.98	6.50	14.07	29.87	51.31
0.0200	2.89	3.70	5.11	12.43	27.90	48.93
0.0300	2.27	3.00	4.34	11.44	26.57	47.17
0.0400	1.88	2.56	3.83	10.71	25.50	45.66
0.0500	1.60	2.23	3.43	10.07	24.50	44.17
0.0600	1.35	1.93	3.06	9.44	23.41	42.50
0.0700	1.13	1.66	2.71	8.80	22.25	40.66
0.0800	0.95	1.42	2.39	8.16	21.03	38.68
0.0900	0.79	1.22	2.10	7.52	19.76	36.57
0.1000	0.65	1.03	1.84	6.90	18.46	34.37
0.1200	0.44	0.73	1.38	5.71	15.82	29.80
0.1400	0.29	0.51	1.02	4.62	13.23	25.18
0.1600	0.19	0.35	0.74	3.65	10.78	20.73
0.1800	0.12	0.23	0.52	2.81	8.56	16.60
0.2000	0.08	0.15	0.36	2.11	6.61	12.94
0.2500	0.02	0.05	0.13	0.92	3.07	6.14
0.3000	0.01	0.01	0.04	0.34	1.20	2.43
0.3500	0.00	0.00	0.01	0.11	0.39	0.81
0.4000			0.00	0.03	0.11	0.22
0.4500				0.01	0.02	0.05
0.5000				0.00	0.00	0.01
0.6000						0.00

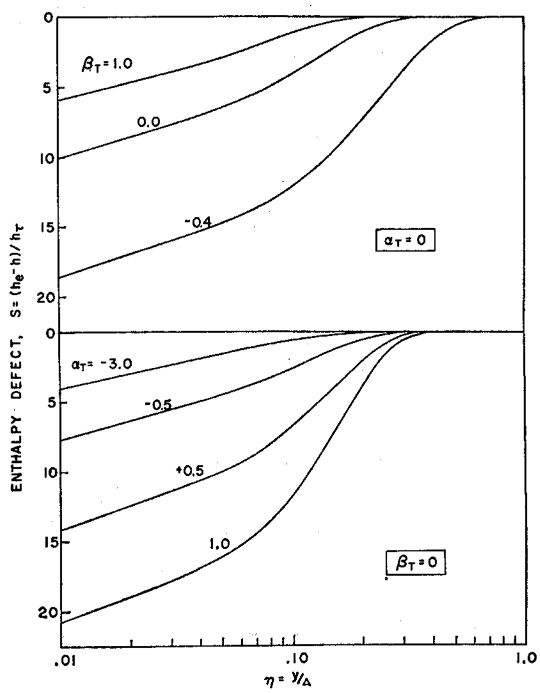

Fig. 2 Incompressible turbulent enthalpy profile defect solutions.

It is noted that if the turbulent Prandtl number is unity, i.e., $\epsilon_h = \epsilon$, and if the wall pressure and temperature are constant, then $\varkappa_h = \varkappa$, $A_h = A$, $B_h = B$, and hence the basic form of Reynolds analogy is recovered, i.e., $C_H = C_f/2$.

The parameter, $c = (\Delta/\gamma_h)(d\gamma_h/dx)$, appearing in Eq. (9) may be obtained by differentiating Eq. (15), and employing the modified form of Reynolds analogy, Eq. (16). A term proportional to the rate of change of the wall temperature appears in this expression. Defining an equilibrium wall temperature parameter $\alpha_T = -\rho u_e\delta^*(dh_w/dx)/q_w$, we find that

$$c = \{\kappa_h\alpha_T + \gamma\beta_T[1 + 1/m][1 + \gamma/\kappa]\}/\{\kappa[1/\gamma - (A + B)] + \kappa_h[A_h + B_h]\} \quad (17)$$

In order to obtain similarity solutions of Eq. (9), we require that β_T, α_T, and γ be constants independent of x.‡

Numerical Solutions and Comparisons with Data

A fourth-order Adams-Moulton numerical integration scheme with a fourth-order Runge-Kutta starter was used to integrate Eq. (9) subject to the boundary conditions of Eq. (10). A Newton-Raphson iteration procedure was employed to determine the correct value of S near the wall, $S(\epsilon)$, $[\epsilon = 10^{-4}]$ such that S vanishes as $\eta \to \infty$. A simultaneous solution of Mellor and Gibson's momentum equation [Eq. (14) of Ref. 3] using the above numerical scheme was also obtained in order to provide appropriate values of $f'(\eta)$ for the energy equation solution. Numerical solutions for $S(\eta)$ were obtained for $R_{\delta^*} = 10^5$ and values of α_T and β_T in the range: $-3 < \alpha_T < 2.5$, $-.4 < \beta_T < 10$. These solutions are tabulated in Tables 1 and 2 and they are shown plotted in Fig. 2.

The value of the outer wake eddy conductivity constant, K_h, was determined by comparing several solutions for the enthalpy profile with the constant pressure outer wake data of Reynolds, Kays, and Kline[8] at two different Reynolds numbers (see Fig. 1). A value of $K_h = 0.018$ appears to give

‡ Mellor and Gibson pointed out that neglecting the variation of f' with γ introduces an error of less than 0.1% in the solutions for the velocity defect profile. A similarly small error is expected in the enthalpy profile solution.

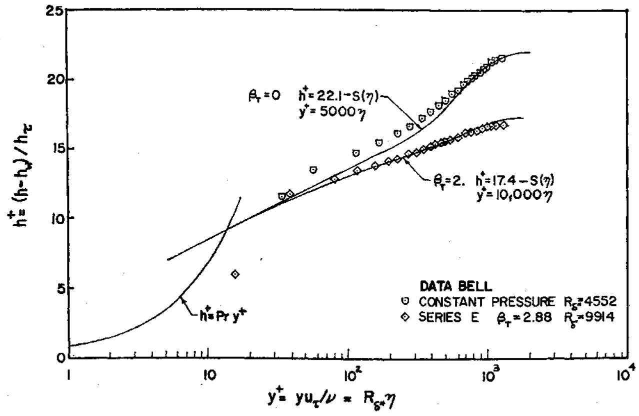

Fig. 3 Effect of pressure gradient parameter, β_T, on turbulent enthalpy profiles, including the laminar sublayer. Comparison with Bell's data (Ref. 9).

the best fit. This constant, along with those previously determined ($\varkappa_h$ and B_h), was used in the calculation of all the equilibrium enthalpy solutions.

In Fig. 3, theoretical solutions for the enthalpy defect parameter $S = (he - h)/h_\tau$, for the cases $\beta = 0$ and $\beta = 2$, are favorably compared with the data of Bell[9] for both constant pressure and adverse pressure gradient flows. In Bell's Series E experiment, boundary layer surveys of total pressure and temperature were made on a heated flat plate in a moderate positive pressure gradient. While the pressure gradient parameter β_T increased along the plate, through the Series E run, β_T only varied between 0.9 and 2.88, in a distance of about 30 boundary layer thicknesses. Thus the average experimental value of β_T up to the $x = 3.275$-ft. station was approximately 2.0. The comparison in Fig. 3 is between a $\beta_T = 2.0$ theoretical solution and the measured profile at the 3.275-ft. station. The comparison with the Series E data illustrates two basic features of enthalpy profiles in adverse pressure gradients which are readily given by the theory. First, the wake portion of the $\beta_T = 2$ enthalpy profile has a negative strength as compared with the $\beta_T = 0$ solution; i.e., the value of h^+ in the outer layer is below that given by the enthalpy law of the wall, Eq. (13). Second, the outer defect solution only matches with the law of the wall solution at a small value of y^+ (≈ 15), approaching the joining point with the laminar sublayer solution. Thus in large adverse pressure gradient flows a law of the wall region for enthalpy profiles will be difficult, if not impossible, to find.

The effect of pressure gradient on the turbulent heat flux distribution is shown in Fig. 4a for the case of constant wall temperature ($\alpha_T = 0$); $q/q_w = -\Phi_h S'$ is plotted vs the similarity distance parameter, η. While the heat flux profiles have a nearly identical shape for the case of $\alpha_T = 0$, (see Fig. 4a) it is demonstrated in Fig. 4b that wall temperature

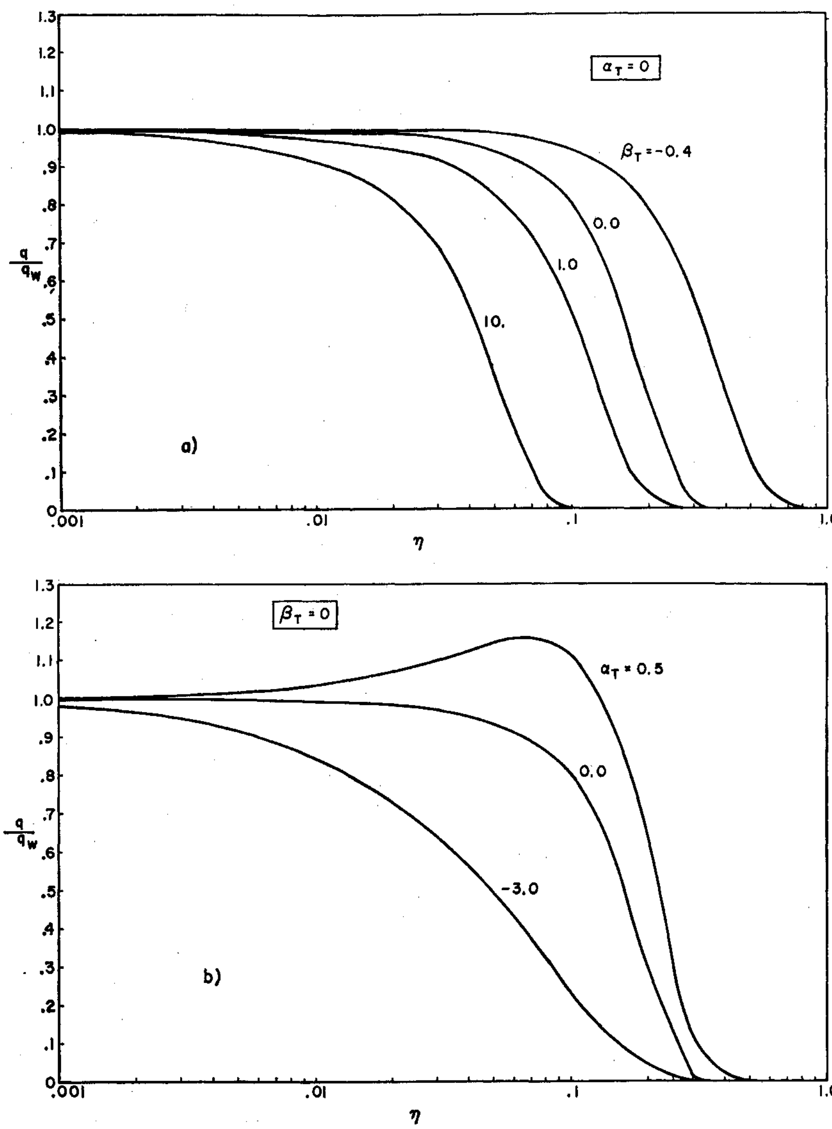

Fig. 4 Turbulent heat flux distributions.

variations can have an important effect on the boundary layer heat flux distribution.

For the case $\alpha_T = 0$, q/q_w remains nearly equal to unity for values of $\eta < 0.03$. Positive values of α_T correspond to the case of decreasing wall temperatures (provided that $h_w > h_e$). Temperatures away from the wall tend to be higher than for the constant temperature case, thus decreasing the temperature gradients near the wall and hence decreasing the wall heat flux. Away from the wall, the temperature gradients are more like those of the constant wall temperature case, hence the heat flux away from the wall tends to be higher than that at the wall. Thus, q/q_w can reach a maximum at some distance η away from the wall, as is shown for the case of $\alpha_T = 0.5$. Negative values of α_T correspond to wall temperatures increasing with x; hence higher wall heat fluxes relative to the constant wall temperature case. Because the heat fluxes in the central part of the layer are nearly equal to those for the $\alpha_T = 0$ case, q/q_w drops off more rapidly than for the $\alpha_T = 0$ problem.

These wall heat transfer trends are made more evident in Fig. 5 where the Reynolds analogy parameter, ($\mathcal{R} = 2C_H/C_f$), given by Eq. (16), is plotted as a function of the wall temperature parameter, α_T. One notes that as α_T increases, $\mathcal{R}$ decreases, indicating a drop in wall heat transfer. In fact, if α_T is increased to a large enough value, C_H tends toward zero. This phenomenon is analogous to the effect of increasing β_T, and noting that the skin friction coefficient, C_f, goes to zero.

Also shown in Fig. 5 is the effect of the pressure gradient parameter, β_T on $\mathcal{R}$. One finds that $\mathcal{R} \to \infty$ as $\beta \to \infty$, indicating that while the skin friction may vanish, C_H remains finite. A careful note should be made of the strong effect of pressure gradient on the Reynolds analogy parameter. If one were to use the assumption that $\mathcal{R} \approx 1.1$ for all pressure gradient flows, it is evident from Fig. 5 that one might overestimate C_H by as much as 25% for expanding flows, and underestimate C_H by 50–60% for moderate equilibrium compressive flows.

Values of Stanton number and skin friction coefficient deduced from the heated adverse pressure gradient measurements of Bell are combined to produce the Reynolds analogy parameter data plotted in Fig. 5 as a function of β_T. The initial flow entering Bell's adverse gradient test region was a constant pressure turbulent boundary layer. Since β_T was rapidly increased to a value of 0.89 at the first measuring station, the value of $\mathcal{R}$ tended to remain at first at the flat plate value. However, with the continued increase in β_T, the Reynolds analogy parameter quickly increased. It is seen that the data closely follows the curve given by the theory for a family of locally similar flows.

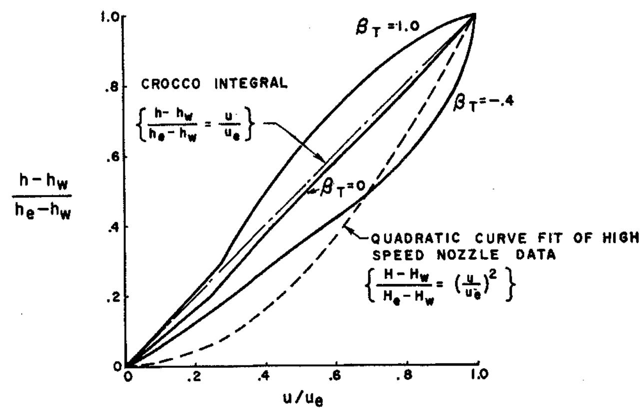

Fig. 6 Effect of pressure gradient parameter on incompressible turbulent enthalpy profiles; comparison with trend of supersonic expansive flow data.[1–2]

Tetervin[5] has recently performed an approximate calculation of the effect of pressure gradient on $\mathcal{R}$, in which the turbulent shear and heat flux profiles throughout the boundary layer are represented by simple polynomials. His approximate solution (with the Colburn correction for Prandtl number) is compared with the present similarity solution in Fig. 5. One notes that both the approximate and similarity solutions yield nearly equal values for $\mathcal{R}$ in the case of flat plate flows, but that Tetervin's solution is seen to predict somewhat larger values of $\mathcal{R}$ than those given by the similarity solution when $\beta_T > 0$. Tetervin has indicated that his solutions may be partially in error due to his assumption of equal thermal and velocity boundary layer thickness ($\delta_h = \delta$). The present similarity solutions indicate that $\delta_h/\delta \approx 0.7$ for $\beta_T \lesssim 3.0$.

Some evidence of the inapplicability of the Crocco temperature relation, $(h - h_w)/(h_e - h_w) = u/u_e$, for nonconstant pressure flows is seen by examination of Fig. 6, where $(h - h_w)/(h_e - h_w)$ (determined from the similarity solutions) is plotted against the velocity ratio u/u_e. Note that in these coordinates the laminar sublayer represents a significant portion of the $h - u$ profile; approximated by an inner linear region of slope $\mathcal{R}Pr$ which extends out to a u/u_e of about 0.3. For the case $\beta_T = 0$, the calculated profile agrees fairly well with the Crocco relation. On the other hand, when $\beta_T < 0$ (corresponding to an expanding flow, $dP/dx < 0$), the calculated enthalpy profile falls rapidly below the Crocco solution. This is the same behavior that experimentalists[1,2] have found in high-speed flows along nozzle walls when boundary layer total enthalpy data is plotted in the form $(H - H_w)/(H_e - H_w)$ vs u/u_e. It was found that the trend of the data can be approximated by a quadratic formula of the form $(H - H_w)/(H_e - H_w) = (u/u_e)^2$. A plot of this equation is presented in Fig. 6 for comparison with the low-speed solution. Although it is not possible to directly relate the low-speed solution to the supersonic domain, it is evident that pressure gradients can play an important effect on boundary layer enthalpy profiles. Even if measurements of enthalpy profiles are made in a region where the pressure gradient is locally equal to zero, the effect of upstream "mean" flow history can maintain the profiles at their $dP/dx < 0$ shapes for a considerable distance downstream of the end of the expansion region. This effect is similar to that noted in turbulent velocity profiles when the pressure gradient is suddenly changed to zero. The velocity profiles retain their upstream character for a distance equal to many boundary layer thicknesses downstream of the beginning of the $dP/dx = 0$ regime

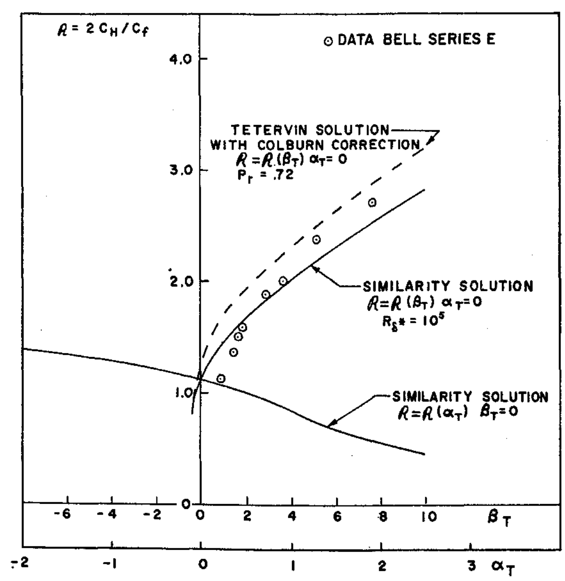

Fig. 5 Effect of wall temperature and pressure gradients on incompressible turbulent Reynolds analogy factor. Comparison with Bell's Series E data.[9]

and only slowly relax to their flat plate shape. The velocity relaxation effect can be calculated using an integral method (e.g., Ref. 10) in which the velocity profiles are unhooked from the local pressure gradient. Likewise, the effect of relaxing enthalpy profiles can be calculated in a similar manner by "unhooking" the enthalpy profiles from the local value of β_T (or α_T) and making them a function of an appropriate shape parameter, e.g.,

$$I_2 = \int_0^\infty S d\eta$$

Conclusions

A family of incompressible equilibrium enthalpy profiles were obtained by integrating the thermal energy equation using a Prandtl-Clauser-like eddy conductivity model ($\kappa_h = 0.45$, $K_h = 0.018$). By matching the outer defect solution with the enthalpy law of the wall ($B_h = 3.35$), an analytical expression was obtained for a local heat transfer law, which allows the calculation of the Reynolds analogy factor for similar (or locally similar) flows for nonzero pressure and wall temperature gradients. The good comparison of the enthalpy and heat transfer data of Bell with the present theory indicates that adverse pressure gradient flows have negative wakes and only a small region of overlap of the outer defect solution with the enthalpy law of the wall. The Reynolds analogy factor $\mathcal{R}$ is shown to increase rapidly for flows with increasing values of the pressure gradient parameter β_T. It was further shown that strong favorable pressure gradients can produce the same deviation of h vs u profiles from the Crocco integral as found by experimentalists in high-speed flow along nozzle walls.

Finally, the two parameter enthalpy profile family, presented herein, in conjunction with the integrated thermal energy equation, and the local heat transfer law, Eq. (16), will make it relatively easy to solve for the development of the heat transfer coefficient, C_H, subject to arbitrary wall temperature and pressure variations in incompressible turbulent flow.

References

[1] Bushnell, D. et al., "Comparison of Prediction Methods and Studies of Relaxation in Hypersonic Turbulent Boundary Layers," TN D-5433, Sept. 1969, NASA.

[2] Bertram, M., Cary, A., and Whitehead, A., "Experiments with Hypersonic Turbulent Boundary Layers on Flat Plates and Wings," *Proceedings of the AGARD Specialists Meeting on Hypersonic Boundary Layers and Flow Fields*, London, May 1968.

[3] Mellor, G. and Gibson, D., "Equilibrium Turbulent Boundary Layers," *Journal of Fluid Mechanics*, Vol. 24, Pt. 2, 1966, pp. 225–253.

[4] Hudimoto, B., "Heat Transfer in a Turbulent Boundary Layer with Pressure Gradient," *Memoirs of the Faculty of Engineering*, Vol. XXV, Kyoto Univ., Kyoto, Japan, 1963.

[5] Tetervin, N., "Approximate Calculation of Reynolds Analogy for Turbulent Boundary Layer with Pressure Gradient," *AIAA Journal*, Vol. 7, No. 6, June 1969, pp. 1079–1085.

[6] Cebeci, T., Smith, A. M. O., and Mosinskis, G., "Solution of the Incompressible Turbulent Boundary Layer Equations with Heat Transfer," Eleventh National Heat Transfer Conference, Minneapolis, Minn., Aug. 3–6, 1969.

[7] Herring, H. J. and Mellor, G. L., "A Method of Calculating Compressible Turbulent Boundary Layers," CR-1144, Sept. 1968, NASA.

[8] Reynolds, W., Kays, W., and Kline, S., "Heat Transfer in the Incompressible Boundary Layer I—Constant Wall Temperature," Memo 12-1-58w, Dec. 1958, NASA.

[9] Bell, J. B., "Heat Transfer to Turbulent Boundary Layers in Pressure Gradients," M.S. thesis, 1966, Univ. of Melbourne. The tabulated data is also presented in *Computation of Turbulent Boundary Layers—1968 AFOSR-IFP-Stanford Conference*, Vol. II, Stanford Univ., Thermosciences Div., 1969.

[10] Alber, I. E., "Application of an Exact Expression for the Equilibrium Dissipation Integral to the Calculation of Turbulent Nonequilibrium Flows," *Proceedings of Turbulent Boundary Layers AFOSR-IFP-Stanford Conference*, Vol. I, edited by S. J. Kline et al., Thermosciences Div., Dept. of Mechanical Engineering, Stanford Univ.